Linear Algebra Review

Outline

1- Matrix Operations
2- Vectors and Normalization
3- Dot product (similarity)
4- Cross product (direxn.s ⊥ to plane)
5- Orthonormal basis
6- Orthogonal Matrix Properties

References
- Notes by Rob Jagnow & Patrick Nichols
- Videos by Ravi Ramamoorthi.
- Notes by Michael Eckman

1- Matrix Operations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$ Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$ Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$ Multiplication (MM).

Notes.
- $AB \neq BA$
- $$[\ \ ]_{N \times K} [\ \ ]_{K \times M} = [\ \ ]_{N \times M}$$
  must match for legal MM.
- Transformations are represented by matrices in Computer Graphics.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$ Transpose.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Identity

$$AI = IA = A$$

$$AA^{-1} = A^{-1}A = I.$$

Identity & Inverses

$$(AB)^{-1} = B^{-1}A^{-1}$$

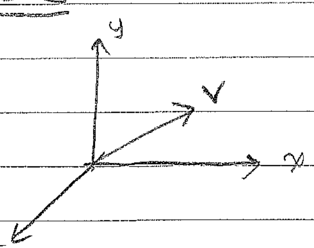
$$(AB)^T = B^T A^T$$

transposing products of matrices

Vector Operations

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

sometimes written as $\vec{v}$



default: column vector

$$\|v\| = \sqrt{a^2 + b^2 + c^2}$$

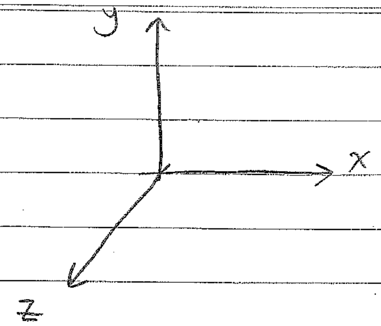
Magnitude

$$= \sqrt{v \cdot v}$$

(dot product, see later)

$$\hat{v} = \frac{v}{\|v\|}$$

Normalization



Right handed std. coord. system

Dot Product

aka: Inner Product

aka: Vector Scalar Product

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

vectors must be of the same length!

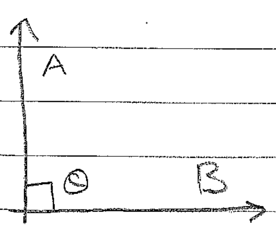
$$A \bullet B = A^T B$$

$$A \bullet (B + C) = A \bullet B + A \bullet C$$

$$(kA) \bullet B = A \bullet (kB) = k(A \bullet B)$$ where k is a scalar.

$$A \bullet B = B \bullet A = \|A\| \, \|B\| \cos \theta$$

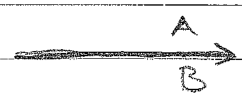
Assuming A and B are unit vectors:



$A \bullet B = 0$

$\therefore \cos 90^\circ = 0$

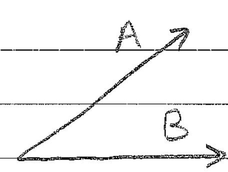
NOT SIMILAR



$A \bullet B = 1$

$\therefore \cos 0^\circ = 1$

MAX SIMILARITY



$A \bullet B = \cos \theta$

SOME SIMILARITY

$\| B \rightarrow A \|$

Magnitude of the projection of B on A

$$\| B \rightarrow A \| = \|B\| \cos \Theta$$

$$= \frac{A \cdot B}{\|A\|}$$

Outer Product

$A \otimes B = AB^T$.

aka: Vector Matrix Product.

$A \times B$

Cross Product

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

a vector orthogonal to both A and B ∴ very useful in creating a co-ordinate system.

$$A \times B = -B \times A$$

$$A \times A = \vec{\emptyset}$$

$$\| A \times B \| = \|A\| \|B\| \sin \Theta$$

$$A \times (B + C) = A \times B + A \times C$$

$A \times (kB) = k(A \times B)$ where k is a constant

3 vectors in 3D : u, v, and n
such that :

$\boxed{\text{Orthonormal Basis}}$

$\uparrow$

orthogonal
+
normal

(1) $\|u\| = \|v\| = \|w\| = 1$

(2) $u \cdot v = v \cdot w = u \cdot w = 0$

(3) $w = u \times v$, $v \times w = u$, $w \times u = v$

Example : $u = (1, 0, 0)$
$v = (0, 1, 0)$
$w = (0, 0, 1)$

This means that we can find an orthonormal basis given any 2 vectors A and B (not necessarily unit norm, not necessarily orthogonal) s.t. $A \neq B$. The following steps demonstrate this :

$$w = \frac{A}{\|A\|}$$

$$u = \frac{B \times w}{\|B \times w\|}$$

$$v = w \times u$$

Orthogonal Matrix

- Each row has length 1, and rows are mutually perpendicular. (columns are also orthonormal basis)
- Example: rotation matrix
  $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Orthogonal Matrix Properties

- The inverse of an orthogonal Matrix is its transpose.
- $$\begin{bmatrix} | & | & | \\ u & v & w \\ | & | & | \end{bmatrix} \quad \begin{matrix} u \cdot u = 1 & u \cdot v = 0 \\ v \cdot v = 1 & u \cdot w = 0 \\ w \cdot w = 1 & v \cdot w = 0 \end{matrix}$$
- $$\begin{bmatrix} — & n & — \\ — & r & — \\ — & s & — \end{bmatrix} \quad \begin{matrix} n \cdot n = 1 & n \cdot r = 0 \\ r \cdot r = 1 & n \cdot s = 0 \\ s \cdot s = 1 & r \cdot s = 0 \end{matrix}$$
- If $AA^T \neq I$, this implies that A is <u>NOT</u> an orthogonal matrix.
- Computing the transpose is much cheaper than computing the inverse.

General Notes

- Transforming a point is a matrix vector multiplication. $\begin{bmatrix} \text{trans.} \end{bmatrix} \begin{bmatrix} \text{pt.} \end{bmatrix}$
- $$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nm} \end{bmatrix}^{-1} = \begin{bmatrix} 1/a_{11} & & & \\ & 1/a_{22} & & \\ & & \ddots & \\ & & & 1/a_{nm} \end{bmatrix}.$$

$\sin^2\theta + \cos^2\theta = 1$